

Harmonic maps between generalized Lagrange spaces

Mircea Neagu

Abstract

In Section 1 the author defines the notion of harmonic map between generalized Lagrange spaces. Section 2 analyses the particular case when the generalized Lagrange spaces are Lagrange spaces of electrodynamics. In Section 3 it is proved that for certain systems of differential or partial differential equations, the solutions are harmonic maps between certain generalized Lagrange spaces, in the sense of Section 1. Section 4 describes the main properties of the generalized Lagrange spaces constructed in Section 3.

Mathematics Subject Classification: 53C60, 49N45, 35R30

Key words: generalized Lagrange spaces, harmonic maps, geodesics, differential equations, partial differential equations.

1 Introduction

Let $(M^m, g_{\alpha\beta})$ and (N^n, h_{ij}) be two generalized Lagrange spaces, where m , respectively n , is the dimension of M , respectively N . The manifold M , respectively N , is coordinated by (a^α) , respectively (x^i) . On $M \times N$, the first m coordinates are indexed by $\alpha, \beta, \gamma, \dots$ and the last n coordinates are indexed by i, j, k, \dots . The fundamental metric tensors are expressed locally by

i) $g_{\alpha\beta} = g_{\alpha\beta}(a, b)$, $\forall \alpha, \beta = \overline{1, m}$, where $(a, b) = (a^\mu, b^\mu)$ are adapted coordinates on TM .

ii) $h_{ij} = h_{ij}(x, y)$, $\forall i, j = \overline{1, n}$, where $(x, y) = (x^k, y^k)$ are adapted coordinates on TN .

On $M \times N$, we consider an arbitrary tensor of type $(1, 2)$, denoted by P , with all components null except $P_{\alpha i}^\beta(a, x)$ and $P_{\alpha i}^j(a, x)$, where $\alpha, \beta = \overline{1, m}$, $i, j = \overline{1, n}$, which will be called *tensor of connection*. The connection tensor P allows one to build the directions b and y of the metric tensors $g_{\alpha\beta}(a, b)$ and $h_{ij}(x, y)$ used in the construction of the energy functional E whose extremals will be the harmonic maps between the generalized Lagrange spaces M and N . In conclusion, this tensor makes the connection between the metric structures of the spaces M , respectively N , and the harmonic maps that we will define.

We assume that the manifold M is connected, compact, orientable and endowed with a Riemannian metric $\varphi_{\alpha\beta}$. These conditions assure the existence of a volume element and, implicitly, of a theory of integration on M . Using the generalized Lagrange metrics $g_{\alpha\beta}$ and h_{ij} and the tensor of connection P , we can define the following

$\begin{pmatrix} P \\ g & \varphi & h \end{pmatrix}$ -energy functional,

$$E = E_{g\varphi h}^P : C^\infty(M, N) \rightarrow R,$$

$$E_{g\varphi h}^P(f) = \frac{1}{2} \int_M g^{\alpha\beta}(a, b(a, f^k, f_\gamma^k)) h_{ij}(f(a), y(a, f^k, f_\gamma^k)) f_\alpha^i f_\beta^j \sqrt{\varphi} da,$$

$$\text{where } \begin{cases} f^i = x^i(f), f_\alpha^i = \frac{\partial f^i}{\partial a^\alpha}, \varphi = \det(\varphi_{\alpha\beta}), \\ b(a, f^k, f_\gamma^k) = b^\gamma(a) \frac{\partial}{\partial a^\gamma} \Big|_a \stackrel{\text{def}}{=} \varphi^{\alpha\beta}(a) f_\alpha^i(a) P_{\beta i}^\gamma(a, f(a)) \frac{\partial}{\partial a^\gamma} \Big|_a, \\ y(a, f^k, f_\gamma^k) = y^k(a) \frac{\partial}{\partial x^k} \Big|_{f(a)} \stackrel{\text{def}}{=} \varphi^{\alpha\beta}(a) f_\alpha^i(a) P_{\beta i}^k(a, f(a)) \frac{\partial}{\partial x^k} \Big|_{f(a)}. \end{cases}$$

Definition. A map $f \in C^\infty(M, N)$ is $\begin{pmatrix} P \\ g & \varphi & h \end{pmatrix}$ -harmonic iff f is a critical point for the functional $E_{g\varphi h}^P$.

Particular cases. i) If $g_{\alpha\beta}(a, b) = \varphi_{\alpha\beta}(a)$ and $h_{ij}(x, y) = h_{ij}(x)$ are Riemannian metrics and the connection tensor is an arbitrary one, it recovers the classical definition of a harmonic map between two Riemannian manifolds [1]. We remark that, in this case, the definition of harmonic maps is independent of the connection tensor field P .

ii) If we take $N = R$, $h_{11} = 1$ and the tensor of connection is of the form $P = (\delta_\beta^\alpha, P_{\beta 1}^1)$, we obtain $C^\infty(M, N) = \mathcal{F}(M)$ and the energy functional becomes

$$E_{g\varphi 1}^P(f) = \frac{1}{2} \int_M g^{\alpha\beta}(a, \text{grad}_\varphi f) f_\alpha f_\beta \sqrt{\varphi} da, \quad \forall f \in \mathcal{F}(M).$$

iii) If we consider $M = [a, b] \subset R$, $\varphi_{11} = g_{11} = 1$ and the tensor of connection is $P = (P_{1i}^1, \delta_i^k)$, we obtain $C^\infty(M, N) = \{x : [a, b] \rightarrow N \mid x - C^\infty \text{ differentiable}\}$. Denoting $C^\infty(M, N) = \Omega_{a,b}(N)$, the energy functional should be

$$E_{11h}^P(x) = \frac{1}{2} \int_a^b h_{ij}(x(t), \dot{x}(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} dt, \quad \forall x \in \Omega_{a,b}(N).$$

In conclusion, the $\begin{pmatrix} P \\ 1 & 1 & h \end{pmatrix}$ -harmonic curves are exactly the geodesics of the generalized Lagrange space $(N, h_{ij}(x, y))$ [2].

2 Harmonic maps between Lagrange spaces of electrodynamics

Let (M^m, L_M) and (N^n, L_N) be Lagrange spaces with the Lagrangians

$$L_M(a, b) = g_{\alpha\beta}(a) b^\alpha b^\beta + g_{\alpha\beta}(a) U^\alpha(a) b^\beta + F(a),$$

$$L_N(x, y) = h_{ij}(x) y^i y^j + h_{ij}(x) V^i(x) y^j + G(x),$$

where

- $g_{\alpha\beta}$ (resp. h_{ij}) is Riemannian metric on M (resp. N) representing the gravitational potentials on M (resp. N).
- U^α (resp. V^i) is a vector field on M (resp. N) representing the electromagnetic potentials.
- F (resp. G) is a smooth function on M (resp. N) representing the potential function.

The fundamental metric tensors of these Lagrangians are

$$g_{\alpha\beta}(a) = \frac{\partial^2 L_M}{\partial b^\alpha \partial b^\beta}, \quad h_{ij}(x) = \frac{\partial^2 L_N}{\partial y^i \partial y^j}.$$

Taking an arbitrary tensor of connection P , the energy functional becomes

$$E_{g\varphi h}(f) = \frac{1}{2} \int_M g^{\alpha\beta}(a) h_{ij}(f(a)) f_\alpha^i f_\beta^j \sqrt{\varphi} \, da.$$

We remark that, in this case, the energy functional is independent of the connection tensor P .

The Euler-Lagrange equations will be, obviously, the equations of harmonic maps, namely

$$g^{\alpha\beta} \left\{ f_{\alpha\beta}^k - \left[G_{\alpha\beta}^\gamma + \frac{1}{2} \frac{\partial}{\partial a^\alpha} \left(\ln \frac{g}{\varphi} \right) \delta_\beta^\gamma \right] f_\gamma^k + H_{ij}^k f_\alpha^i f_\beta^j \right\} = 0, \quad \forall k = \overline{1, n},$$

where

- $f_{\alpha\beta}^k = \frac{\partial^2 f^k}{\partial a^\alpha \partial a^\beta}$, $g = \det(g_{\alpha\beta})$, $\varphi = \det(\varphi_{\alpha\beta})$.
- $G_{\alpha\beta}^\gamma$ are the Christoffel symbols of the metric $g_{\alpha\beta}$.
- H_{ij}^k are the Christoffel symbols of the metric h_{ij} .

Remarks. i) If $g_{\alpha\beta} = \varphi_{\alpha\beta}$, we recover the classical equations of harmonic maps between two Riemannian manifolds.

ii) Denoting $\Delta_{\alpha\beta}^\gamma = G_{\alpha\beta}^\gamma + \frac{1}{2} \frac{\partial}{\partial a^\alpha} \left(\ln \frac{g}{\varphi} \right) \delta_\beta^\gamma$, we remark that $\Delta_{\alpha\beta}^\gamma$ represent the components of a linear connection induced by the metrics $g_{\alpha\beta}$ and $\varphi_{\alpha\beta}$.

By the last remark, we can give the following

Definition. Let g, φ be Riemannian metrics on M . The curve $c : I \rightarrow M$, expressed locally by $c(t) = (a^\alpha(t))$, is a (g, φ) -geodesic iff c is an autoparallel curve of the connection $\Delta_{\alpha\beta}^\gamma$ induced by the metrics g and φ , namely

$$\frac{d^2 a^\gamma}{dt^2} = -\Delta_{\alpha\beta}^\gamma \frac{da^\alpha}{dt} \frac{da^\beta}{dt}.$$

Remarks. i) If $g = \varphi$, then we recover the classical definition of a geodesic on the Riemannian manifold $(M, g = \varphi)$.

ii) It is obviously that a (g, φ) -geodesic is a reparametrized geodesic of the metric g .

Theorem. Let $f : (M, L_M) \rightarrow (N, L_N)$ be a smooth map which carries (g, φ) -geodesics into h -geodesics. Then f is harmonic map.

Proof. Let $c : I \subset \mathbb{R} \rightarrow M$, $c(t) = (a^\alpha(t))$ be a (g, φ) -geodesic. Then we have $\frac{d^2 a^\gamma}{dt^2} = -\Delta_{\alpha\beta}^\gamma \frac{da^\alpha}{dt} \frac{da^\beta}{dt}$. Because $\bar{c}(t) = f(c(t))$ is h -geodesic, it follows that $\frac{d^2 \bar{c}^k}{dt^2} + H_{ij}^k \frac{d\bar{c}^i}{dt} \frac{d\bar{c}^j}{dt} = 0$. But $\frac{d\bar{c}^k}{dt} = f_\alpha^k(c(t)) \frac{da^\alpha}{dt} \Rightarrow \frac{d^2 \bar{c}^k}{dt^2} = f_{\alpha\beta}^k \frac{da^\alpha}{dt} \frac{da^\beta}{dt} + f_\alpha^k \frac{d^2 a^\alpha}{dt^2}$. In conclusion we obtain

$$\begin{aligned} \frac{d^2 a^\alpha}{dt^2} f_\alpha^k + f_{\alpha\beta}^k \frac{da^\alpha}{dt} \frac{da^\beta}{dt} + H_{ij}^k f_\alpha^i f_\beta^j \frac{da^\alpha}{dt} \frac{da^\beta}{dt} &= 0 \Rightarrow \\ \left(f_{\alpha\beta}^k - \Delta_{\alpha\beta}^\gamma f_\gamma^k + H_{ij}^k f_\alpha^i f_\beta^j \right) \frac{da^\alpha}{dt} \frac{da^\beta}{dt} &= 0, \forall k = \overline{1, n}. \quad \square. \end{aligned}$$

3 Geometrical interpretation of solutions of certain PDEs of order one

The problem of finding a geometrical structure of Riemannian type on a manifold M such that the orbits of an arbitrary vector field X should be geodesics, was intensively studied by Sasaki. The results were not satisfactory, but, in his study, Sasaki discovered the well known almost contact structures on a manifold of odd dimension [6]. After the introduction of generalized Lagrange spaces by Miron [2], the same problem is resumed by Udriște [8, 9]. This succeeded to discover a Lagrange structure on M , depending of the vector field X and an associated $(1,1)$ -tensor field, such that the orbits of C^2 class should be geodesics. Moreover, he formulated a more general problem [9], namely

1) Are there structures of Lagrange type such that the solutions of certain PDEs of order one should be *harmonic maps*?

2) What is a *harmonic map* between two generalized Lagrange spaces?

A partial answer of these questions is offered by author in his paper [4], using the notion of harmonic map on a direction between a Riemannian manifold and a generalized Lagrange manifold. The notion of *harmonic map* between two generalized Lagrange spaces introduced in [7] allows one to extend the results of previous papers [4], [8], [9] and to obtain a beautiful geometrical interpretation of the solutions of the certain PDEs of order one.

For every smooth map $f \in C^\infty(M, N)$, we use the following notation

$$\delta f = f_\alpha^i da^\alpha|_a \otimes \frac{\partial}{\partial y^i} \Big|_{f(a)} \in \Gamma(T^*M \otimes f^{-1}(TN)).$$

On $M \times N$, let T be one tensor of type $(1, 1)$ with all components equal to zero except $(T_\alpha^i)_{\substack{i=\overline{1, n} \\ \alpha=\overline{1, m}}}$. Let the system of partial differential equations

$$(E) \quad \delta f = T \text{ expressed locally by } \frac{\partial f^i}{\partial a^\alpha} = T_\alpha^i(a, f(a)).$$

If $(M, \varphi_{\alpha\beta})$ and (N, ψ_{ij}) are Riemannian manifolds, we can build a scalar product on $\Gamma(T^*M \otimes f^{-1}(TN))$ by $\langle T, S \rangle = \varphi^{\alpha\beta}(a) \psi_{ij}(f(a)) T_\alpha^i S_\beta^j$, where $T = T_\alpha^i da^\alpha \otimes \frac{\partial}{\partial y^i}$ and $S = S_\beta^j da^\beta \otimes \frac{\partial}{\partial y^j}$.

Under these conditions, we can prove the following

Theorem. *If $(M, \varphi), (N, \psi)$ are Riemannian manifolds and $f \in C^\infty(M, N)$ is a solution of the system (E), then f is a solution of the variational problem associated to the functional $\mathcal{L}_T : C^\infty(M, N) \setminus \{f \mid \exists a \in M \text{ such that } \langle \delta f, T \rangle(a) = 0\} \rightarrow R_+$,*

$$\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\|\delta f\|^2 \|T\|^2}{\langle \delta f, T \rangle^2} \sqrt{\varphi} da = \frac{1}{2} \int_M \frac{\|T\|^2}{\langle \delta f, T \rangle^2} \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i f_\beta^j \sqrt{\varphi} da.$$

Proof. In the space $\Gamma(T^*M \times f^{-1}(TN))$, the Cauchy inequality for the scalar product \langle, \rangle holds. It follows that the following inequality is true, $\langle T, S \rangle^2 \leq \|T\|^2 \|S\|^2$, $\forall T, S \in \Gamma(T^*M \times f^{-1}(TN))$, with equality if and only if there exists $\mathcal{K} \in \mathcal{F}(M)$ such that $T = \mathcal{K}S$. Consequently, for every $f \in C^\infty(M, N)$ we have

$$\mathcal{L}(f) = \frac{1}{2} \int_M \frac{\|\delta f\|^2 \|T\|^2}{\langle \delta f, T \rangle^2} \sqrt{\varphi} da \geq \frac{1}{2} \int_M \sqrt{\varphi} da = \frac{1}{2} Vol_\varphi(M).$$

Obviously, if f is a solution of the system (E), we obtain $\mathcal{L}_T(f) = \frac{1}{2} Vol_\varphi(M)$, that is, f is a global minimum point for \mathcal{L}_T \square .

Remarks. i) In certain particular cases of the system (E), the functional \mathcal{L}_T becomes exactly a functional of type $\begin{pmatrix} P \\ g & \varphi & h \end{pmatrix}$ -energy.

ii) The global minimum points of the functional \mathcal{L}_T are solutions of the system $\delta f = \mathcal{K}T$, where $\mathcal{K} \in \mathcal{F}(M)$, not necessarily with $\mathcal{K} = 1$.

iii) Replacing the Riemannian metric ψ_{ij} by a pseudo-Riemannian metric, the preceding theorem survives because the form of the Euler-Lagrange equations remains unchanged. The difference is that the solutions of the system (E), in the pseudo-Riemannian case, are not the global minimum points for the functional \mathcal{L}_T . Moreover, the statement (ii), of above, does not hold.

Fundamental examples.

1. Orbits

For $M = ([a, b], 1)$ and $T = \xi \in \Gamma(x^{-1}(TN))$, the system (E) becomes

$$(E_1) \quad \frac{dx^i}{dt} = \xi^i(x(t)), \quad x : [a, b] \rightarrow N,$$

that is the system of orbits for ξ , and the functional \mathcal{L}_ξ is

$$\mathcal{L}_\xi(x) = \frac{1}{2} \int_a^b \frac{\|\xi\|_\psi^2}{[\xi^b(x)]^2} \psi_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} dt,$$

where $\xi^b = \xi_i dx^i = \psi_{ij} \xi^j dx^i$. Hence the functional \mathcal{L}_ξ is a $\begin{pmatrix} P \\ 1 & 1 & h \end{pmatrix}$ -energy (see (iii) of first particular cases of this paper), where

$$h_{ij} : TN \setminus \{y \mid \xi^b(y) = 0 \text{ for some } y\} \rightarrow R$$

is defined by

$$h_{ij}(x, y) = \frac{\|\xi\|_\psi^2}{[\xi^b(y)]^2} \psi_{ij}(x) = \psi_{ij}(x) \exp \left[2 \ln \frac{\|\xi\|_\psi}{[\xi^b(y)]} \right].$$

This case is studied in other way by Udriște in [8]-[9].

2. Pfaffian systems

For $N = (R, 1)$ and $T = A \in \Lambda^1(T^*M)$, the system (E) becomes

$$(E_2) \quad df = A, \quad f \in \mathcal{F}(M),$$

that is a Pfaffian system, and the functional \mathcal{L}_T reduces to

$$\mathcal{L}_A(f) = \frac{1}{2} \int_M \frac{\|A\|_\varphi^2}{[A(\text{grad}_\varphi f)]^2} \varphi^{\alpha\beta} f_\alpha f_\beta \sqrt{\varphi} da.$$

Hence, the functional \mathcal{L}_A is a $\begin{pmatrix} P \\ g & \varphi & 1 \end{pmatrix}$ -energy (see (ii) of first particular cases of this paper), where $g_{\alpha\beta} : TM \setminus \{b \mid A(b) = 0 \text{ for some } b\} \rightarrow R$ is defined by

$$g_{\alpha\beta}(a, b) = \frac{[A(b)]^2}{\|A\|_\varphi^2} \varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a) \exp \left[2 \ln \frac{|A(b)|}{\|A\|_\varphi} \right].$$

3. Pseudolinear functions

We suppose that $T_\beta^k(a, x) = \xi^k(x) A_\beta(a)$, where ξ^k is vector field on N and A_β is 1-form on M . In this case the system (E) is

$$(E_3) \quad \frac{\partial f^k}{\partial a^\beta} = \xi^k(f) A_\beta(a)$$

and the functional \mathcal{L}_T is expressed by

$$\begin{aligned} \mathcal{L}_T(f) &= \frac{1}{2} \int_M \frac{\|\xi\|_\psi^2 \|A\|_\varphi^2}{[A(b)]^2} \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i f_\beta^j \sqrt{\varphi} da = \\ &= \frac{1}{2} \int_M g^{\alpha\beta}(a, b) h_{ij}(f(a)) f_\alpha^i f_\beta^j \sqrt{\varphi} da, \end{aligned}$$

where $h_{ij}(x) = \|\xi\|_\psi^2 \psi_{ij}(x)$, the tensor of connection is $P_{i\beta}^\gamma(x) = \delta_\beta^\gamma \xi_i(x)$, $b^\gamma = \varphi^{\alpha\beta} f_\alpha^i P_{i\beta}^\gamma$ and the Lagrange metric tensor

$$g_{\alpha\beta} : TM \setminus \{b \mid A(b) = 0 \text{ for some } b\} \rightarrow R$$

is defined by

$$g_{\alpha\beta}(a, b) = \frac{[A(b)]^2}{\|A\|_\varphi^2} \varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a) \exp \left[2 \ln \frac{|A(b)|}{\|A\|_\varphi} \right].$$

It follows that the functional \mathcal{L}_T becomes a $\begin{pmatrix} P \\ g & \varphi & h \end{pmatrix}$ -energy.

Remark. Take M to be an open subset in $(R^n, \varphi = \delta)$ and $N = (R, \psi = 1)$, the system from the third example is

$$(PL) \quad \frac{\partial f}{\partial a^\alpha} = \xi(a) A_\alpha(f(a)), \quad \forall \alpha = \overline{1, m}.$$

Supposing that $(grad f)(a) \neq 0, \forall a \in M$, the solutions of this system are the well known *pseudolinear functions* [5]. These functions have the following property,

–for every fixed point $x_0 \in M$, the hypersurface of constant level

$$M_{f(x_0)} = \{x \in M \mid f(x) = f(x_0)\}$$

is totally geodesic [5] (i. e. the second fundamental form vanishes identically).

In conclusion, the pseudolinear functions are examples of harmonic maps between the generalized Lagrange spaces $\left(M, g_{\alpha\beta}(a, b) = \frac{[A(b)]^2}{\|A\|^2} \delta_{\alpha\beta}\right)$ and $(R, h(x) = \xi^2(x))$. For example, we have the following pseudolinear functions [5]:

3. 1. $f(a) = e^{<v, a>+w}$, where $v \in M, w \in R$, is solution for the system (PL) with $\xi(a) = 1$ and $A(f(a)) = f(a)v$.

3. 2. $f(a) = \frac{<v, a>+w}{<v', a'>+w'}$, where $v, v' \in M, w, w' \in R$, is solution for (PL) with $\xi(a) = \frac{1}{<v', a>+w'}$ and $A(f(a)) = v - f(a)w$.

Remark. The preceding cases appear also in [7] and, from another point of view, in [4]. The following case is the main novelty of this paper.

4. The general case

If we have $T_\alpha^i(a, x) = \sum_{r=1}^t \xi_r^i(x) A_\alpha^r(a)$, where $\{\xi_r\}_{r=\overline{1, t}} \subset \mathcal{X}(N)$ is a family of vector fields on N and $\{A^r\}_{r=\overline{1, t}} \subset \Lambda^1(T^*M)$ is a family of 1-forms on M , the system of equations (E) reduces to

$$(E_4) \quad \frac{\partial f^i}{\partial a^\alpha} = \sum_{r=1}^t \xi_r^i(f) A_\alpha^r(a).$$

Without loss of generality, we can suppose that $\{\xi_r\}_{r=\overline{1, t}} \subset \mathcal{X}(N)$ (resp. $\{A^r\}_{r=\overline{1, t}} \subset \Lambda^1(T^*M)$) are linearly independent. In these conditions, we shall have $t \leq \min\{m, n\}$, where $m = \dim M$ and $n = \dim N$.

4. 1. Assume that $\{\xi_r\}_{r=\overline{1, t}} \subset \mathcal{X}(N)$ is an orthonormal system of vector fields with respect to the Riemannian metric ψ_{ij} on N . Let $B \in \Lambda^1(T^*M)$ be an arbitrary unit 1-form on M . With our assumptions, by a simple calculation, we obtain

$$\begin{aligned} \|T\|^2 &= \varphi^{\alpha\beta} \psi_{ij} \xi_r^i A_\alpha^r \xi_s^j A_\beta^s = \sum_{r,s=1}^t <\xi_r, \xi_s>_\psi <A^r, A^s>_\varphi = \sum_{r=1}^t \|A^r\|_\varphi^2, \\ <\delta f, T> &= \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i \xi_r^j A_\beta^r B^\mu B_\mu. \end{aligned}$$

Defining the tensor of connection by $P_{i\beta}^\gamma(a, x) = \psi_{ij}(x) \xi_r^j(x) A_\beta^r(a) B^\gamma(a)$ and $b^\gamma = \varphi^{\alpha\beta} f_\alpha^i P_{i\beta}^\gamma$, the functional \mathcal{L}_T takes the form

$$\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\sum_{r=1}^t \|A^r\|_\varphi^2}{[B(b)]^2} \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i f_\beta^j \sqrt{\varphi} da = \frac{1}{2} \int_M g^{\alpha\beta}(a, b) \psi_{ij}(f(a)) f_\alpha^i f_\beta^j \sqrt{\varphi} da,$$

where the Lagrange metric tensor $g_{\alpha\beta} : TM \setminus \{b \mid B(b) = 0 \text{ for some } b\} \rightarrow R$ is expressed by

$$g_{\alpha\beta}(a, b) = \frac{[B(b)]^2}{\sum_{r=1}^t \|A^r\|_\varphi^2} \varphi_{\alpha\beta}(a) = \varphi_{\alpha\beta}(a) \exp \left[2 \ln \frac{|B(b)|}{\sqrt{\sum_{r=1}^t \|A^r\|_\varphi^2}} \right].$$

Consequently, the functional \mathcal{L}_T is a $\begin{pmatrix} P \\ g & \varphi & \psi \end{pmatrix}$ -energy.

4. 2. As above, we assume that the system $\{A^r\}_{r=\overline{1,t}}$ of 1-forms is orthonormal with respect to the metric $\varphi^{\alpha\beta}$ and we choose an arbitrary unit vector field $X \in \mathcal{X}(N)$. By analogy to **4. 1** we shall have

$$\|T\|^2 = \sum_{r=1}^t \|\xi_r\|_\psi^2 \text{ and } \langle \delta f, T \rangle = \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i \xi_r^j A_\beta^r X^k X_k.$$

Using the notations $P_{i\beta}^k(a, x) = \psi_{ij}(x) \xi_r^j(x) A_\beta^r(a) X^k(x)$ and $y^k = \varphi^{\alpha\beta} f_\alpha^i P_{i\beta}^k$ we obtain the following expression of the functional \mathcal{L}_T ,

$$\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\sum_{r=1}^t \|\xi_r\|_\psi^2}{[X^b(y)]^2} \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i f_\beta^j \sqrt{\varphi} da = \frac{1}{2} \int_M \varphi^{\alpha\beta}(a) h_{ij}(f(a), y) f_\alpha^i f_\beta^j \sqrt{\varphi} da,$$

where the Lagrange metric tensor $h_{ij} : TN \setminus \{y \mid X^b(y) = 0 \text{ for some } y\} \rightarrow R$ is

$$h_{ij}(x, y) = \frac{\sum_{r=1}^t \|\xi_r\|_\psi^2}{[X^b(y)]^2} \psi_{ij}(x) = \psi_{ij}(x) \exp \left[2 \ln \frac{\sqrt{\sum_{r=1}^t \|\xi_r\|_\psi^2}}{|X^b(y)|} \right].$$

Obviously, the functional \mathcal{L}_T is a $\begin{pmatrix} P \\ \varphi & \varphi & h \end{pmatrix}$ -energy.

Remarks. i) For the use of the above, we assume *a priori* a Riemannian metric φ or ψ on M or N such that the system of covectors $\{A^r\}_{r=\overline{1,t}}$ or of vectors $\{\xi_r\}_{r=\overline{1,t}}$ is orthonormal. This fact is always possible. In conclusion, our assumptions on the orthonormality of these systems do not restrict the generality of problem.

ii) In the particular case, $n = r = 1$, $N = R$, $\psi_{11} = 1$, $\xi = \frac{d}{dx}$, and $A \in \Lambda^1(T^*M)$ is an arbitrary 1-form on M , we recover the Pfaffian system $df = A$. In this situation, taking $B \in \Lambda^1(T^*M)$ to be an arbitrary unit 1-form, we obtain, for the functional \mathcal{L}_T , the expression

$$\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\|A\|^2}{[B(b)]^2} \varphi^{\alpha\beta} f_\alpha f_\beta \sqrt{\varphi} da = \frac{1}{2} \int_M \frac{\|A\|^2}{[A(\text{grad}_\varphi f)]^2} \varphi^{\alpha\beta} f_\alpha f_\beta \sqrt{\varphi} da,$$

where the tensor of connection is defined by $P_{1\beta}^\gamma = A_\beta B^\gamma$ and the direction b is $b^\gamma = \varphi^{\alpha\beta} f_\alpha^i A_\beta B^\gamma = B^\gamma A(\text{grad}_\varphi f)$.

Since the last integral is the functional \mathcal{L}_A from the example **1**, we remark that the solutions of the Pfaffian system $df = A$ can be regarded in an infinity manner as $\begin{pmatrix} P \\ g & \varphi & 1 \end{pmatrix}$ -harmonic maps. This fact appears because the tensor of connection P

and the generalized Lagrange metric g are dependent of the arbitrary unit covector field B .

iii) Analog to ii), if we take $m = r = 1$, $M = [a, b]$, $g_{11} = \varphi_{11} = 1$, $A = dt$ and $\xi \in \mathcal{X}(N)$ is an arbitrary vector field on N , we find the system of orbits for ξ , that is,

$$\frac{dx^i}{dt} = \xi^i(x(t)), \quad x : [a, b] \rightarrow N.$$

Starting with $X \in \mathcal{X}(N)$ an arbitrary unit vector field, the functional \mathcal{L}_T becomes

$$\mathcal{L}_T(x) = \frac{1}{2} \int_a^b \frac{\|\xi\|_\psi^2}{[X^b(y)]^2} \psi_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} dt = \frac{1}{2} \int_a^b \frac{\|\xi\|_\psi^2}{[\xi^b(\dot{x})]^2} \psi_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} dt,$$

where the tensor of connection is $P_{i1}^k = \xi_i^b X^k$ and $y^k = \xi^b(\dot{x}) X^k$. Because we can vary the tensor of connection P and the generalized Lagrange metric h by the arbitrary unit vector field X , we remark that the trajectories of the vector field ξ can be also regarded in an infinity manner as $\begin{pmatrix} P & & \\ 1 & 1 & h \end{pmatrix}$ -harmonic maps.

4 Lagrange geometry asociated to PDEs of order one

We first remark that, in all above cases, the solutions of C^2 class of the system $\delta f = T$ becomes harmonic maps between generalized Lagrange spaces, in the sense defined in this paper. Moreover, the above generalized Lagrange structures are of type $(M^n, e^{2\sigma(x,y)} \gamma_{ij}(x))$, where $\sigma : TM \setminus \{\text{Hyperplane}\} \rightarrow R$ is a smooth function. Using the ideas exposed in [2], in these spaces, we can construct a Lagrange geometry and field theory. This geometrical Lagrange theory will be regarded as a natural one associated to the PDE system $\delta f = T$, in the sense of the first Udriște's question.

Now, we assume that a generalized Lagrange space $(M^n, g_{ij}(x, y))$ satisfies the following axioms:

- a. 1. The fundamental tensor field $g_{ij}(x, y)$ is of the form

$$g_{ij}(x, y) = e^{2\sigma(x,y)} \gamma_{ij}(x).$$

- a. 2. The space is endowed with the non-linear connection

$$N_j^i(x, y) = \Gamma_{jk}^i(x) y^k,$$

where $\Gamma_{jk}^i(x)$ are the Christoffel symbols for the Riemannian metric $\gamma_{ij}(x)$.

Under these assumptions, our space verifies a constructive-axiomatic formulation of General Relativity due to Ehlers, Pirani and Schild [2]. This space represents a convenient relativistic model, since it has the same conformal and projective properties as the Riemannian space (M, γ_{ij}) .

In the Lagrangian theory of electromagnetism, the electromagnetic tensors F_{ij} and f_{ij} are

$$F_{ij} = \left(g_{ip} \frac{\delta \sigma}{\delta x^j} - g_{jp} \frac{\delta \sigma}{\delta x^i} \right) y^p, \quad f_{ij} = \left(g_{ip} \frac{\partial \sigma}{\partial y^j} - g_{jp} \frac{\partial \sigma}{\partial y^i} \right) y^p.$$

Developping the formalism presented in [2], [3] and denoting by r_{jkl}^i the curvature tensor field of the metric $\gamma_{ij}(x)$, the following Maxwell equations of the electromagnetic tensors hold

$$\begin{cases} F_{ij|k} + F_{jk|i} + F_{ki|j} = - \sum_{(ijk)} g_{ip} r_{qjk}^h \frac{\partial \sigma}{\partial y^h} y^p y^q, \\ F_{ij|k} + F_{jk|i} + F_{ki|j} = -(f_{ij|k} + f_{jk|i} + f_{ki|j}), \\ f_{ij|k} + f_{jk|i} + f_{ki|j} = 0, \end{cases}$$

where $|_i$ (resp. $|_a$) represents the h - (resp. v -) covariant derivative induced by the non-linear connection N_j^i .

Suppose $\sigma = \sigma(x)$. Then the v - electromagnetic tensor is $f_{ij} = 0$, the h - covariant operator " $|_i$ " becomes the covariant derivative with respect to Levi-Civita connection of the metric $g_{ij}(x, y) = e^{2\sigma(x)} \varphi_{ij}(x)$, the h - electromagnetic tensor F_{ij} is the same with the classical electromagnetic tensor and the Maxwell's equations reduce to the classical ones.

In the construction of the gravitational field equations, we shall use the notations

$$\begin{cases} r_{ij} = r_{ijk}^k, \quad r = \gamma^{ij} r_{ij}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \\ \sigma^H = \gamma^{kl} \frac{\delta \sigma}{\delta x^k} \frac{\delta \sigma}{\delta x^l}, \quad \sigma^V = \gamma^{ab} \frac{\partial \sigma}{\partial y^a} \frac{\partial \sigma}{\partial y^b}, \quad \bar{\sigma} = \gamma^{ij} \sigma_{ij}, \quad \dot{\sigma} = \gamma^{ab} \dot{\sigma}_{ab}, \end{cases}$$

where $\sigma_{ij} = \frac{\delta \sigma}{\delta x^i} |_{|j} + \frac{\delta \sigma}{\delta x^i} \frac{\delta \sigma}{\delta x^j} - \frac{1}{2} \gamma_{ij} \sigma^H$, $\dot{\sigma}_{ab} = \frac{\partial \sigma}{\partial y^a} \Big|_b + \frac{\partial \sigma}{\partial y^a} \frac{\partial \sigma}{\partial y^b} - \frac{1}{2} \gamma_{ab} \sigma^V$.

In these conditions, the Einstein's equations of the space $(M, g_{ij}(x, y))$ take the form

$$\begin{cases} r_{ij} - \frac{1}{2} r \gamma_{ij} + t_{ij} = \mathcal{K} T_{ij}^H \\ (2 - n)(\dot{\sigma}_{ab} - \dot{\sigma} \gamma_{ab}) = \mathcal{K} T_{ab}^V, \end{cases}$$

where T_{ij}^H and T_{ab}^V are the h - and the v - components of the energy momentum tensor field, \mathcal{K} is the gravific constant and

$$t_{ij} = (n - 2)(\gamma_{ij} \bar{\sigma} - \sigma_{ij}) + \gamma_{ij} r_{st} y^s \gamma^{tp} \frac{\partial \sigma}{\partial y^p} + \frac{\partial \sigma}{\partial y^i} r_{tja}^a y^t - \gamma_{is} \gamma^{ap} \frac{\partial \sigma}{\partial y^p} r_{tja}^s y^t.$$

It is clear that the general metric $g_{ij}(x, y) = e^{2\sigma(x, y)} \varphi_{ij}(x)$ implies Einstein equations which differ from the classical ones by the additional tensor t_{ij} .

Finally, we remark that, in certain particular cases, it is posible to build a generalized Lagrange geometry and field theory naturally attached to a system of partial differential equations. In these geometrical structures, the solutions of C^2 class of the PDE system become harmonic maps. This idea was suggested by Udriște in private discussions and in [9], [10].

Open problem. Because the generalized Lagrange structure constructed in this paper is not unique, it arises a natural question:

–Is it possible to build a unique generalized Lagrange geometry naturally asociated to a given PDEs system?

An answer to this question will be offered by author in a subsequent paper, using a more general Lagrange geometry, naturally attached to a multidimensional Lagrangian defined on the jet fibration of order one.

Acknowledgements. I would like to express my gratitude to the reviewer of Southeast Asian Bulletin of Mathematics and Prof. Dr. C. Udriște for their valuable comments and very useful suggestions.

References

- [1] J. Eells, L. Lemaire, *A Report on Harmonic Maps*, Bull. London Math. Soc. 10 (1978), 1-68.
- [2] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.
- [3] R. Miron, R. K. Tavakol, V. Balan, I. Roxburgh, *Geometry of Space-Time and Generalized Lagrange Gauge Theory*, Publ. Math. , Debrecen, Hungary, 42, 3-4 (1992), 215-224.
- [4] M. Neagu, *Solutions of Inverse Problems for Variational Calculus*, Workshop on Diff. Geom. , Global Analysis, Lie Algebras, Aristotle University of Thessaloniki, Greece, June 24-28, 1998; BSG Proceedings 4, pp 180-187, Editor:Prof. Dr. Grigorios Tsagas, Geometry Balkan Press, Bucharest, 2000.
- [5] T. Rapcsák, *Smooth Nonlinear Optimization in R^n* , Kluwer Academic Publishers, 1997.
- [6] S. Sasaki, *Almost Contact Manifolds, I, II, III*, Mathematical Institute Tohoku University, 1965, 1967, 1968.
- [7] C. Udriște, M. Neagu, *Geometrical Interpretation of Solutions of Certain PDEs*, Balkan Journal of Geometry and Its Applications, 4, 1 (1999), 145-152.
- [8] C. Udriște, A. Udriște, *Electromagnetic Dynamical Systems*, Balkan Journal of Geometry and Its Applications, 2, 1 (1997), 129-140.
- [9] C. Udriște, *Geometric Dynamics*, Second Conference of Balkan Society of Geometers, Aristotle University of Thessaloniki, Greece, June 23-26, 1998; Southeast Asian Bulletin of Mathematics, Springer-Verlag, 24 (2000), 1-11.
- [10] C. Udriște, *Nonclassical Dynamics*, Proceedings of Conference on Mathematics in honour of Professor Radu Roșca at the occasion of his ninetieth birthday, Katholieke Universiteit Brussel, Katholieke Universiteit Leuven, Dec. 11-16, 1999, to appear.

University POLITEHNICA of Bucharest
 Department of Mathematics I
 Splaiul Independentei 313
 77206 Bucharest, Romania
 e-mail:mircea@mathem.pub.ro